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Degenerate representations of $\mathbf{GL}_n$ over a p-adic field

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(joint work with David Helm)

Let F be a finite extension of $\mathbb{Q}_p$ and let $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ be the category of smooth complex representations of $\mathrm{GL}_n(F)$ for some integer $n \geq 1$. In this talk I explained how one can define a stratification of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ by using the theory of degenerate Whittaker models.

Degenerate Whittaker spaces. Let U_n be the unipotent radical of the standard Borel subgroup of $\mathrm{GL}_n(F)$ and fix a nontrivial additive character $\psi\colon F \to \mathbb{C}^\times$. For any integer partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ of n, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$, we can define a character of U_n in the following way. Let S_λ be the subset of $\{1, 2, \dots, n\}$ consisting of integers that cannot be expressed in the form $\lambda_1 + \dots + \lambda_s$ for some $s \leq r$. We then define the character $\psi_\lambda\colon U_n \to \mathbb{C}^\times$ to be

$$\psi_\lambda(u) = \sum_{i \in S_\lambda} \psi(u_{i,i+1}).$$

The *degenerate Whittaker space* associated to λ is the representation

$$W_\lambda = \text{c-Ind}_{U_n}^{\mathrm{GL}_n(F)}(\psi_\lambda).$$

Note that for the partition (n) of n, we have that $W_{(n)}$ is the Gelfand–Graev representation and an irreducible representation is generic if and only if it admits a nontrivial map from $W_{(n)}$. For an arbitrary irreducible representation $\pi \in$

$\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ it is then natural to ask if there are partitions λ of n such that there is a nontrivial map from W_λ to π. This can be answered by using the theory of Bernstein–Zelevinsky derivatives.

Highest derivative partition. Recall for each integer $1 \leq i \leq n$ the i-th Bernstein–Zelevinsky derivative ([1, Section 4]), which is an exact functor

$$(-)^{(i)}\colon \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)) \to \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_{n-i}(F)).$$

The *highest derivative* of a representation $\pi \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ is the largest integer k such that the k-th Bernstein–Zelevinsky derivative $\pi^{(k)}$ is nonzero. If π is irreducible with highest derivative k, Zelevinsky showed that $\pi^{(k)}$ is an irreducible representation of $\mathrm{GL}_{n-k}(F)$. For any irreducible representation π we can hence define a sequence of numbers $(\lambda_1, \lambda_2, \dots)$, where λ_i is the highest derivative of

$$((\pi^{(\lambda_1)})^{(\lambda_2)\cdots})^{(\lambda_{i-1})}.$$

Zelevinsky ([4, Theorem 8.1]) proved that these numbers form an integer partition of n, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots$ and $\lambda_1 + \lambda_2 + \cdots = n$, which we call the *highest derivative partition* of π.

The highest derivative partition is an important invariant of an irreducible representation and equals the partition associated to the maximal member of the wave front set of the representation. Moreover, it can be interpreted under the local Langlands correspondence ([3, Theorem B]).

By using this notion, Mœglin and Waldspurger proved that, if an irreducible representation π has highest derivative partition $\lambda = (\lambda_1, \lambda_2, \dots)$, then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{GL}_n(F)}(W_\lambda, \pi) = 1.$$

Moreover, if

$$\mathrm{Hom}_{\mathrm{GL}_n(F)}(W_{\lambda'}, \pi) \neq 0$$

for some partition $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ of n, then λ' precedes λ under the dominance order on partitions, i.e.

$$\sum_{i=1}^{j} \lambda'_i \leq \sum_{i=1}^{j} \lambda_i$$

for all $j \geq 1$.

A Stratification of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. Let e be a primitive idempotent in the Bernstein center $\mathfrak{Z}_n$ of $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and consider the associated Bernstein block $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. For any partition λ of n, we consider the following full subcategories of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$:

- $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\preceq\lambda}$, whose objects are representations in $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ for which every irreducible subquotient has highest derivative partition λ' satisfying $\lambda' \preceq \lambda$.
- $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\prec\lambda}$, a full subcategory of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{\preceq\lambda}$, with the additional condition that no irreducible subquotient has highest derivative partition λ.

Let $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ be the Serre quotient

$$e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda} \Big/ e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\prec\lambda}.$$

We will now construct a progenerator for $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$. From the projectivity of the Gelfand–Graev representation it follows that the W_λ are projective objects in $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$. Moreover, the functor

$$\begin{aligned} e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)) &\to e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda} \\ \pi \mapsto \pi^{\preceq\lambda} &= \mathrm{coker}\left(\oplus_{\lambda'\npreceq\lambda} W'_{\lambda'} \otimes \mathrm{Hom}_{G_n}(W'_{\lambda'},\pi) \to \pi\right) \end{aligned}$$

is left adjoint to the natural inclusion of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F)))^{\preceq\lambda}$ into $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and hence right exact. This together with the results of Mœglin and Waldspurger mentioned above imply that $(eW_\lambda)^{\preceq\lambda}$ is a progenerator of $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$.

Let eE_λ be the endomorphism ring of $(eW_\lambda)^{\preceq\lambda}$. The upshot of the above is that we have an equivalence of categories between $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))^{=\lambda}$ and the category of right eE_λ-modules. Hence it is natural to study the rings eE_λ and we are able to prove the following result.

Theorem [2]**.** For any primitive idempotent e of $\mathfrak{Z}_n$ and integer partition λ of n the ring eE_λ is commutative and reduced.

Furthermore, we are able to describe the rings eE_λ explicitly, similiar to the description of the Bernstein center due to Bernstein–Deligne.

Example. For the partition (n) we have that $(eW_{(n)})^{\preceq(n)} = eW_{(n)}$ is the component of the Gelfand–Graev representation in $e\,\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$ and it is a well known result that its endomorphism ring $eE_{(n)}$ is isomorphic to the commutative and reduced ring $e\mathfrak{Z}_n$.

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